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The ^{4}He trimer as an Efimov system*†

Dedicated to the 40th anniversary of the Efimov effect

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Abstract We review the results obtained in the last four decades which demonstrate the Efimov nature of the ^{4}He three-atomic system.

Keywords Efimov effect · helium trimer · three-body problem

1 Introduction

For many years the existence of a bound state of two ^{4}He atoms was an open problem. Some potential models predicted such a state [1, 2, 3] while the others did not [4, 5]. However practically all more or less realistic helium-helium potentials generated a very large atom-atom scattering length of about 90–100 Å or even more (see, e.g., [6, Table I]). As soon as the Efimov effect has been discovered [7, 8, 9], it was to be expected that the system of three ^{4}He atoms possesses bound states of Efimov type. For the first time this idea has been suggested and substantiated by Lim, Duffy, and Damert [10], just seven years after Efimov's first works on his effect [7, 8]. It is the Efimov effect that distinguishes the ^{4}He atoms from the atoms of all other noble gases, and makes the ^{4}He clusters especially attractive objects of experimental and theoretical studies.

Almost all realistic He-He-potentials constructed in the 1970s and later supported $^4He_2$ binding, although the binding energies may differ by tens of times [2, 11, 12]. The semi-empirical Aziz *et al.* potentials [13, 14] are considered particularly adequate, as well as the purely theoretical TTY potential by Tang, Toennies, and Yiu [15]. Compared to the others, the LM2M2 potential [14] seems to be most often used in ^{4}He trimer calculations of the last decade. Besides these potentials, we also mention the SAPT potentials developed by Korona *et al.* [16], by Janzen and Aziz [17], and by Jeziorska *et al.* [18]. All potentials [13, 14, 15, 16, 17, 18] support a single bound state of two ^{4}He

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atoms with a binding energy of 1.3–1.9 millikelvin (mK). For convenience, we collect in Table 1 the $^4He_2$ binding energies and ^{4}He–^{4}He scattering lengths obtained with the Aziz *et al.* potentials HFD-B [13], LM2M2 [14], and with the TTY potential by Tang, Toennies, and Yiu [15]. Similarly to the SAPT potentials [16,17,18], these three potentials predict exactly two bound states for the ^{4}He trimer. The HFD-B potential gives about 133 mK for the ground state energy [19,20,21,22] and about 2.74 mK for the energy of the excited state [19,20]. The corresponding results for the TTY potential are shown in Table 2. The respective energies obtained for the LM2M2 potential are practically the same as for the TTY potential (see Table 3), that is, they are close to 126 mK for the ground state and close to 2.28 mK for the excited one [19,20,21,22,23,24,25,26,27,28,29,30,31]. For the ^{4}He trimer binding energies obtained with the SAPT potentials we refer to the calculations in [21,30,35,37].

Results on the ^{4}He atom – ^{4}He dimer scattering length and phase shifts with realistic atom-atom potentials are less numerous. In this respect we refer to [19,22,24,30,32,33,34,36,37]. For a discussion of problems of convergence arising in ^{4}He – $^4He_2$ scattering calculations see [38, Section 5]. The ^{4}He atom – ^{4}He dimer scattering lengths available for the TTY and LM2M2 potentials are shown in Tables 2 and 3, respectively.

By now, it is already rather well established that, if the ^{4}He trimer excited state exists, then it should be of Efimov nature. As already mentioned, the appearance of the Efimov effect in the ^{4}He three-atom system was conjectured in [10] where the $^4He_3$ excited state binding energy has been calculated for the first time by means of the Faddeev integral equations. Even more convincing arguments in favor of this phenomenon were presented by Cornelius and Glöckle [39] who also employed the momentum space Faddeev equations. Ten years later the conclusions of [39] were strongly supported in [40] and [33]. The calculations of [40] were based on the adiabatic hyperspherical expansion in three-body configuration space, while the hard-core version of the two-dimensional Faddeev differential equations has been used in [33]. References [41,42,43,44,45,46, 47,48] suggest that the $^4He_3$ ground state itself may be considered as an Efimov state since, given the ^{4}He-^{4}He atom-atom scattering length, both the $^4He_3$ ground-state and excited-state energies lye on the same universal scaling curve (for details, see, e.g., [49, Sections 6.7 and 6.8]).

Experimentally, ^{4}He dimers have been observed for the first time in 1993 by the Minnesota group [50], and in 1994 by Schöllkopf and Toennies [51]. Along with the dimers, the experiment [51] established also the existence of ^{4}He trimers. A first experimental estimate for the size of the $^4He_2$ molecule has been given in [52]. According to this reference, the root mean square distance between ^{4}He nuclei in the ^{4}He dimer is equal to 62 ± 10 Å. Several years later, the bond length for $^4He_2$ was measured again by Grisenti *et al.* [53] who found for this length the value of 52 ± 4 Å. The estimates of [52] and [53] make the ^{4}He dimer the most extended known diatomic molecular ground state. The measurements [53] also allowed to evaluate a ^{4}He-^{4}He scattering length of 104^{+8}_{-18} Å and a ^{4}He dimer energy of $1.1^{+0.3}_{-0.2}$ mK.

Table 1 Calculated values of ^{4}He dimer energy ε_d, bond length $\langle R \rangle$, and ^{4}He atom-atom scattering length $\ell_{\rm sc}^{(2)}$ for three different atom-atom potentials, compared to the corresponding experimental values; estimates for the number $N_{\rm Efi}$ of Efimov states based on formula (4).

Potential	ε_d (mK)	$\ell_{\rm sc}^{(2)}$ (Å)	$\langle R \rangle$ (Å)	$N_{\rm Efi}$[b]
HFD-B	-1.68541	88.50	46.18	0.80
LM2M2	-1.30348	100.23	52.00	0.83
TTY	-1.30962	100.01	51.89	0.83
Experiment[a]	$1.1^{+0.3}_{-0.2}$	104^{+8}_{-18}	52^{+4}_{-4}	

[a]Reference [53]. [b]Reference [6].

Table 2 Ground-state (E_0) and excited-state (E^*) energies of the ^{4}He trimer and the ^{4}He atom-dimer scattering length $\ell_{sc}^{(1+2)}$ in case of the TTY atom-atom potential [15].

	[19]	[56]	[22]	[20]	[26]	[21]	[27]
$\lvert E_0 \rvert$ (mK)	125.8	126.0	126.1	126.40	126.4	126.4	126.2
$\lvert E^* \rvert$ (mK)	2.282^a			2.280		2.277	
$\ell_{sc}^{(1+2)}$ (Å)	116^b			115.8^c			

^{a}This value was rounded in [19]. ^{b}Result of extrapolation (see [36]). ^{c}Result from Ref. [34].

Table 3 Ground-state (E_0) and excited-state (E^*) energies of the ^{4}He trimer and the ^{4}He atom-dimer scattering length $\ell_{sc}^{(1+2)}$ in case of the LM2M2 atom-atom potential [14].

	[28]	[20]	[24]	[19]	[23]	[30]	[29]
$\lvert E_0 \rvert$ (mK)	126.507	126.41	126.39	125.9	126.2	126.15	125.6
$\lvert E^* \rvert$ (mK)	2.276	2.271	2.268	2.282	2.26	2.274	2.245
$\ell_{sc}^{(1+2)}$ (Å)		115.4^a	115.56	115^b	126^c	120.91	

^{a}Result from Ref. [34]. ^{b}Result of extrapolation (see [36]). ^{c}Result from Ref. [22].

In 2000, a promising suggestion has been made by Hegerfeldt and Köhler [54] concerning the experimental observation of an Efimov state in ^{4}He trimers. The suggestion was to study diffraction of ultracold ^{4}He clusters by inclined diffraction gratings and look for the specific traces of the excited trimers in the diffraction picture. The practical realization [55] of such an experiment on a grating of a 1000 Å period did not lead to a convincing success. So, a reliable experimental evidence for the existence of excited states in ^{4}He trimers is still missing. However, in the experiment [55] the size of the ^{4}He$_3$ ground state has been estimated for the first time. According to [55] the He-He bond length in the ^{4}He$_3$ ground state is 11^{+4}_{-5} Å, in agreement with theoretical predictions.

The paper is organized as follows. In Section 2 we recall the basics of the Efimov effect and make historic references to several approaches that were used to prove this phenomenon, including the references to rigorous mathematical proofs. Although this is not directly related to helium trimers, we also give references to recent experimental works on Efimov physics in ultracold alcali-atom gases. In Section 3 we present a computational evidence for the Efimov nature of the ^{3}He trimer excited state, being based on the investigation in [57,58]. Whenever the replacement $V \to \lambda V$ of a realistic atom-atom potential V is being made, this consideration shows how the excited state disappears for some $\lambda > 1$. It is absorbed by the continuous spectrum and turned into a virtual state. For some $\lambda < 1$ an additional excited state pops up, being born from another virtual state. It is this unusual behavior of the energy levels that indicates that the ^{4}He$_3$ excited state originates due to the Efimov effect.

2 Efimov effect

The Efimov effect is a remarkable phenomenon that may be viewed as an excellent illustration for the variety of possibilities arising when we pass from the two-body to the three-body problem. It is well known (see, e.g., [59, Section XIII.3]) that any two-particle system with a sufficiently rapidly decreasing and not too singular interaction $V(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^3$, has a finite number of binding energies. Moreover, the number $\mathfrak{N}(V)$ of these energies, counting multiplicities, satisfies the cele-

brated Birman-Schwinger estimate (see, e.g., [59, Theorem XIII.10])

$$\mathfrak{N}(V) \leq \left(\frac{1}{4\pi}\right)^2 \int\limits_{\mathbb{R}^3} d\boldsymbol{x} \int\limits_{\mathbb{R}^3} d\boldsymbol{y} \frac{|V(\boldsymbol{x})||V(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^2}. \tag{1}$$

Here, it is assumed that the units are chosen in such a way that the two-body Schrödinger operator in the c.m. system reads $(H\psi)(\boldsymbol{x}) = \big(-\Delta_{\boldsymbol{x}} + V(\boldsymbol{x})\big)\psi(\boldsymbol{x})$ with $\boldsymbol{x}$ the reduced Jacobi variable and $\Delta_{\boldsymbol{x}}$ the Laplacian in $\boldsymbol{x}$. Thus, the convergence of the integral on the r.h.s. part of (1) ensures that the number of the two-particle binding energies is finite. In the case of three-particle systems, even with finitely supported smooth two-body potentials, just the opposite statement may be true: under certain conditions the number of binding energies appears to be infinite. Such a spectral situation arises, in particular, for a system of three spinless particles if none of the two-body subsystems has bound states but at least two of them have infinite s-wave scattering lengths. This is the essence of the Efimov effect [7,8]. There is a rigorous mathematical proof that, for the situation described above, the number $N(E)$ of three-body binding energies lying below a value $E < 0$ is increasing logarithmically as $E \to 0$. Moreover, the following limit exists [60] (see also [61])

$$\lim_{E\uparrow 0} \frac{N(E)}{|\ln|E||} = \Upsilon > 0. \tag{2}$$

The value of Υ does not depend on details of the (rapidly decreasing) two-body potentials. It is determined only by the ratios of particle masses. A qualitative analysis, performed by Efimov himself in [7,8,9], allows one to expect that the following limit exists as well

$$\lim_{n\to\infty} \frac{E_{n+1}}{E_n} = \exp\left(-\frac{1}{\Upsilon}\right),$$

where E_n denotes the bound-state energies numbered in the order of decreasing absolute values. Furthermore, if the particles are identical bosons, then Efimov's consideration results in the following asymptotic relationship

$$\lim_{n\to\infty} \frac{E_{n+1}}{E_n} = \exp(-2\pi/\omega_0) \approx \frac{1}{515.035},$$

where $\omega_0 \approx 1.0062378$ is a unique positive solution to the transcendental equation

$$1 - \frac{8}{\sqrt{3}} \frac{\sinh\frac{\pi\omega}{6}}{\omega \cosh\frac{\pi\omega}{2}} = 0. \tag{3}$$

This equation first appeared yet in a work [62] by Danilov on the Skornyakov–Ter-Martirosyan equation [63]. Some rigorous statements on a more detailed asymptotic behavior of the Efimov energy levels E_n, $n \to \infty$, in a system of three identical bosons can be found in [64]. The analysis of [64] follows an alternative approach to justify the Efimov effect, the one that was proposed independently by Faddeev in [65] and Amado and Noble in [66,67] soon after the papers [7,8] were published. This approach involves an explicit separation of a non-Fredholm (as $\ell_{\rm sc}^{(2)} \to \infty$) component of the integral operator entering the momentum-space Faddeev equations and the subsequent examination of the three-body spectrum generated by that component (see [68, pp. 103–105]). Notice that the first completely rigorous proof of the existence of the Efimov effect, given by Yafaev in [69], also follows the approach of [65,66,67]. For rigorous results on Efimov properties of N-body systems with $N \geq 4$ we refer to [70,71,72].

All known two-body systems (both nuclear and atomic) have finite scattering lengths. Therefore, in general it is impossible to observe the genuine "full-scale" Efimov effect (with an infinite number of three-body bound states). Nevertheless, systems featuring at least some peculiarities of the Efimov effect are also of great interest. A qualitative analysis, for the system of three identical bosons, performed by Efimov in [7,8] (see also the review paper by Fillips [73]) shows that if the boson-boson scattering length $\ell_{\rm sc}^{(2)}$ is large compared to the effective radius r_0 of the two-body forces, then

there is an effective $1/\rho^2$-type attractive interaction on a scale of $r_0 \lesssim \rho \lesssim \ell_{\rm sc}^{(2)}$ where ρ is the system hyperradius. This conclusion is used as an argument to approve the following estimate (see [7]):

$$N_{\rm Efi} \simeq \frac{\omega_0}{\pi} \ln \left| \frac{\ell_{\rm sc}^{(2)}}{r_0} \right|, \tag{4}$$

where $N_{\rm Efi}$ denotes the total number of bound states in the three-boson system under consideration. Surely, this estimate is assumed to work only for very large ratios $\ell_{\rm sc}^{(2)}/r_0$ but it may provide a hint also for relatively small $\ell_{\rm sc}^{(2)}/r_0$. Based on (4), for the ^{4}He three-atomic system with realistic atom-atom potentials one typically obtains $N_{\rm Efi} \simeq 0.6$—1.3 (see [6]; cf. Table 1). Since this value is only around (or even less than) unity, it neither supports nor disproves the claim that the excited state of the ^{4}He trimer is a genuine Efimov state, and a further investigation is needed (see Section 3).

We also notice that, if the two-body scattering lengths are infinite, introduction of three-body forces (provided they are short range) does not affect the Efimov effect, because of its long-range nature, and the number of binding energies remains infinite. But if the Efimov effect is not full-scale, i.e. the two-body scattering lengths are large but finite, the appropriately chosen positive definite three-body interation may, of course, completely eliminate any binding in the three-body system.

Already equation (3) drops a hint that there should be a close link between the Efimov effect and the Thomas effect [74]. Recall that the origin of the Thomas effect lies in the fact that, in the case where two-body interactions are zero-range, the three-boson Shrödinger operator is not semi-bounded from below [75]. Hence, there is a possibility of a collapse of the system with all three particles falling to the center of mass. Virtually, the asymptotic estimate (4) explains both the effects at once. For $r_0 \neq 0$ and $|\ell_{\rm sc}^{(2)}| \to \infty$ it gives us the number of Efimov levels, which accumulate exponentially towards the three-body breakup threshold. On the other hand, if $\ell_{\rm sc}^{(2)}$ is finite and nonzero, this estimate describes the number of energy levels in the Thomas effect going to $-\infty$ as $r_0 \to 0$. That the Efimov and Thomas effects are nothing but the two sides of the same coin was noted, in particular, in [76] and [77] (see also [78, Section 5] and references therein). Currently, there are a lot of discussions on the universal properties of three-body systems at ultralow energies, and there is a tendency (see, e.g. [43,79]) to use a joint term “Thomas-Efimov levels” for the discrete spectrum arising in both effects. Various three-body universality aspects and the Efimov effect itself are discussed in great detail in the advanced review article by Braaten and Hammer [49].

Till now we only talked on isolated three-particle systems that do not interact with the rest of the world. It is usually assumed, explicitly or implicitly (see, e.g. [80,81,82,83]), that the estimates like (4) are also valid for three-atom systems put into an external magnetic field. It is known that, being subject to a magnetic field, certain two-atom systems experience a Feshbach resonance due to Zeeman interaction [85]. In such a case one gets an opportunity to control the atom-atom scattering length, by changing the intensity of the magnetic field. This is particularly relevant for systems composed of alkaline atoms. In 2006, the results of an experiment on three-body recombination in an ultracold gas of cesium atoms have been announced [83,86]. Those results were interpreted by the authors of the experiment as evidence of the emergence of at least one Efimov state in the ^{133}Cs three-atomic system as the magnetic field appropriately changes. A discussion and different interpretations of the experiment [83,86] can be found in [79,87]. An experimental evidence for the Efimov resonant states in heteronuclear three-atom systems consisting of ^{41}K and ^{87}Rb atoms was reported in [88]. Recently, signatures of the Efimov effect have been found experimentally in a three-component gas consisting of ^{6}Li atoms that are settled in the three different lowest-energy states [89].

3 On the Efimov nature of the ^{4}He trimer excited state

Although quite different Aziz *et al.* atom-atom potentials [2,13,14] and very different numerical techniques were used in [33,39,40], the main conclusions concerning the trimer excited state are

basically the same. Namely, this state disappears if the potential is multiplied by a factor λ of about 1.2. More precisely, if the atom-atom potential is multiplied by $\lambda > 1$ then the following effect is observed. First, with increasing λ the trimer excited state energy $E^*(\lambda)$ goes deeper more rapidly than the dimer energy $\varepsilon_d(\lambda)$, i.e. the difference $\varepsilon_d(\lambda) - E^*(\lambda)$ increases. At some point the behavior of this difference changes to the opposite, that is, with further increase of λ it decreases monotonously. In other words, from now on the dimer energy $\varepsilon_d(\lambda)$ goes down quicker than the excited-state energy $E^*(\lambda)$. At $\lambda \approx 1.2$ the level E^* disappears, being covered by the continuous spectrum. It is just such a nonstandard behavior of the energy $E^*(\lambda)$ that points to the Efimov nature of the trimer excited state. Vice versa, if λ slightly decreases from 1 by not more than 2 %, the second excited state E^{**} shows up [39,40]. In [24] and [90] the Efimov nature of the ^{4}He trimer excited state was discussed in terms of the atom-atom scattering length.

Apparently, the most detailed numerical study of the nature of the excited state in the ^{4}He trimer has been performed in [58] (see also [91] and [92]). Notice that the Aziz *et al.* potential [13] was employed in [58] and the number of the partial-wave Faddeev components was reduced to one. This, however, should not affect the basic qualitative conclusions. One of the goals of [58] was to elucidate the fate of the excited state, once it leaves the physical sheet (at some $\lambda > 1$). Another goal was to study the emergence mechanism for new excited states as λ ($\lambda < 1$) is decreasing.

It was found in [58] that, for λ ($\lambda > 1$) increasing, the trimer excited-state energy $E^*(\lambda)$ merges with the two-body threshold $\varepsilon_d(\lambda)$ at $\lambda \approx 1.175$. As the factor λ decreases further, it transforms into a first-order virtual level. New excited-state energy levels at $\lambda < 1$ emerge from the first-order virtual levels as well. The latter show up in pairs. The emergence of a pair of first-order virtual levels is preceded by a collision and subsequent fusion of a pair of conjugate first-order resonances into a second-order virtual level. It is worth to notice, however, that these resonances may not be the true resonances, since they are lying outside the energy region where the applicability of the computational approach of [58] was proven to work (see [93]).

Table 4 ^{4}He dimer binding energy ε_d, energies of the first (E^*) and second (E^{**}) excited states of the ^{4}He trimer; virtual-state energy $E_{\rm virt}$ of the ^{4}He three-atom system; ^{4}He atom-atom and ^{4}He atom-dimer scattering lengths $\ell_{\rm sc}^{(2)}$ and $\ell_{\rm sc}^{(1+2)}$, respectively, as functions of the potential strength factor λ. All energies are given in mK, the scattering lengths in Å. The dashes mean the nonexistence of the corresponding states. The HFD-B atom-atom potential [13] was used in the computations.

λ	ε_d	$\varepsilon_d - E^*$	$\varepsilon_d - E_{\rm virt}$	$\varepsilon_d - E^{**}$	$\ell_{\rm sc}^{(1+2)}$	$\ell_{\rm sc}^{(2)}$
1.30	-199.45	-	1.831	-	-61	11.4
1.20	-99.068	-	0.01552	-	-340	14.7
1.18	-82.927	-	0.00058	-	-1783	15.8
1.17	-75.367	0.0063	-	-	8502	16.3
1.15	-61.280	0.0737	-	-	256	17.7
1.10	-32.222	0.4499	-	-	152	23.1
1.0	-1.685	0.773	-	-	160	88.6
0.995	-1.160	0.710	-	-	151	106
0.990	-0.732	0.622	-	-	143	132
0.9875	-0.555	0.573	0.222	-	125	151
0.985	-0.402	0.518	0.097	-	69	177
0.982	-0.251	0.447	0.022	-	-75	223
0.980	-0.170	0.396	0.009	-	-337	271
0.9775	-0.091	0.328	0.003	-	-6972	370
0.975	-0.036	0.259	-	0.002	7120	583
0.973	-0.010	0.204	-	0.006	4260	1092

As an illustration of what has been said above, we present Table 4 taken from [57]. It is seen that for $0.9875 < \lambda \leq 1.17$ the ^{4}He trimer has only one excited state of energy E^* (see the third column). For $\lambda \geq 1.18$, instead of the excited state a virtual state of energy $E_{\rm virt}$ shows up (see the

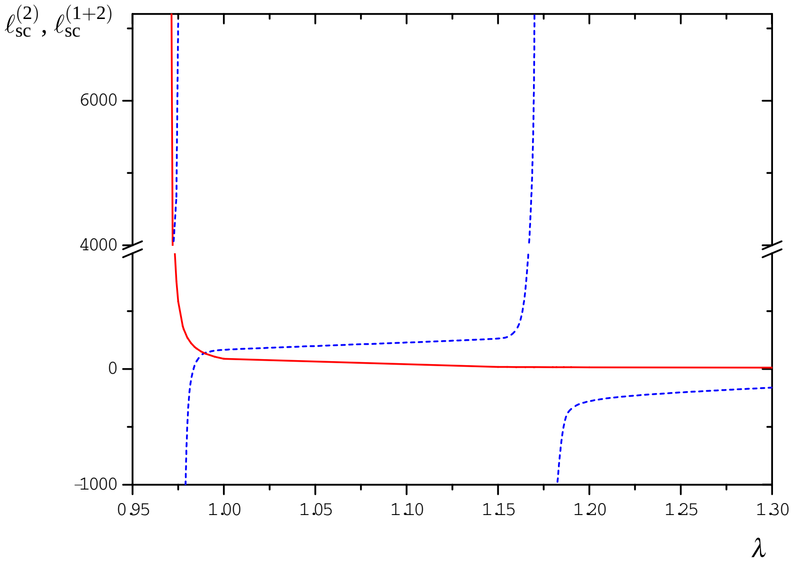

Fig. 1 Dependence of the ^{4}He atom-atom scattering length $\ell_{sc}^{(2)}$ (Å) (solid curve) and the ^{4}He atom-dimer scattering length $\ell_{sc}^{(1+2)}$ (Å) (dashed curve) on the potential strength factor λ. The HFD-B atom-atom potential of Ref. [13] was used in the computations.

fourth column). This occurs as a consequence of the excited-state energy passing to the unphysical sheet.

As λ decreases down to approximately 0.986, a new virtual level arises (see the fourth column again). We use the same notation $E_{\rm virt}$ for the energy of that level. A further decrease of the factor λ to approximately 0.976 shifts the virtual level $E_{\rm virt}$ to the physical sheet, which results in the emergence of the second excited state (see the fifth column). The binding energy of this state is denoted by E^{**}.

In both of the above cases, the transformation of a virtual state into an excited state changes the sign of the atom-dimer scattering length $\ell_{sc}^{(1+2)}$. At the corresponding values of λ the function $\ell_{sc}^{(1+2)}(\lambda)$ has pole-like singularities (see the sixth column of Table 4) while the atom-atom scattering length $\ell_{sc}^{(2)}$ varies continuously and monotonously. The behavior of both the scattering lengths $\ell_{sc}^{(2)}(\lambda)$ and $\ell_{sc}^{(1+2)}$ shown in Table 4 is graphically displayed in Fig. 1.

4 Conclusion

We have reviewed results obtained in the last forty years which prove the Efimov nature of the ^{4}He three-atomic system. This system appears to be the best, most thoroughly investigated example where the Efimov effect manifests itself. According to what is shown, the most vital questions in this context have been asked and answered. There are, of course, numerous other questions that concern, e.g., the Efimov aspects of larger He$_n$ systems, the influence of external fields, the properties of mixed atomic systems etc. All this is the topic of further investigations based on Efimov's fundamental idea (see, e.g., [94,95,96,97] and references therein).

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